Theory Manual for *AeroLS*
Aerodynamic Lifting Surface Analysis Program

Version 1.0.0

by

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License

For more information on AeroLS or to obtain the latest version (software, documentation and source code) visit the following world wide web address:
http://homepage.mac.com/jhuwaldt/java/Applications/AeroLS/AeroLS.html

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1. Introduction

This document describes and partially derives the theory that is implemented by the AeroLS lifting surface analysis program. Understanding this material is critical to proper use of the AeroLS program. It is especially important to have a good understanding of where use of this theory is valid and where it is not.

AeroLS is an aerodynamic lifting surface analysis program. This means that it computes the effects of air flowing around thin (planform and camber only, no thickness) wings which are operating at a small angle of attack and sideslip in flow that is subsonic, steady, inviscid, and irrotational.

AeroLS is part of a large family of analysis codes that are referred to as “linear aero codes” because it assumes that the effects of planform, camber, thickness and friction are linearly separable. This means that these effects can be analyzed separately and then added together to give the total forces acting on the vehicle. AeroLS, at this time, only computes the effects of planform and camber. In reality, all these effects are not linearly separable, but for some cases (subsonic, high Reynolds number flow, on well designed wings at low lift coefficients), this assumption is not such a bad one and can give good 1st order values for educational or quick evaluation purposes.

AeroLS, at this time, makes no attempt to model or make corrections for the effects of leading edge suction, tip vortex roll up, wake distortion, or viscous fluid properties.

The coordinate system used by AeroLS is shown in Figure 1.1. The origin is typically located forward of the configuration being analyzed. The X-coordinate axis runs along the longitudinal length of the vehicle and is positive aft. The Z-coordinate axis runs vertically and is positive up. The Y-coordinate is then found by the right-hand rule and is positive to the right of the vehicle (as viewed from behind looking forward).

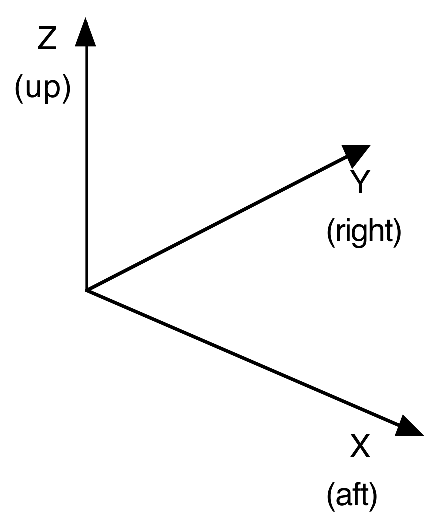

Figure 1.1: Coordinate System used by AeroLS

The approach used by AeroLS is to assume that there is a continuous distribution of bound vorticity over a wing surface which generates the forces acting on that wing. This bound vorticity can then be approximated by a finite number of discrete vortices as shown in Figure 1.2. The individual "horseshoe" vortices are placed in trapezoidal panels which are sometimes called *finite elements* or *lattices*. The procedure used to obtain the numerical solution to the flow is termed a *vortex lattice method* (VLM). However, AeroLS differs in some details from many classical VLM's found in the literature.

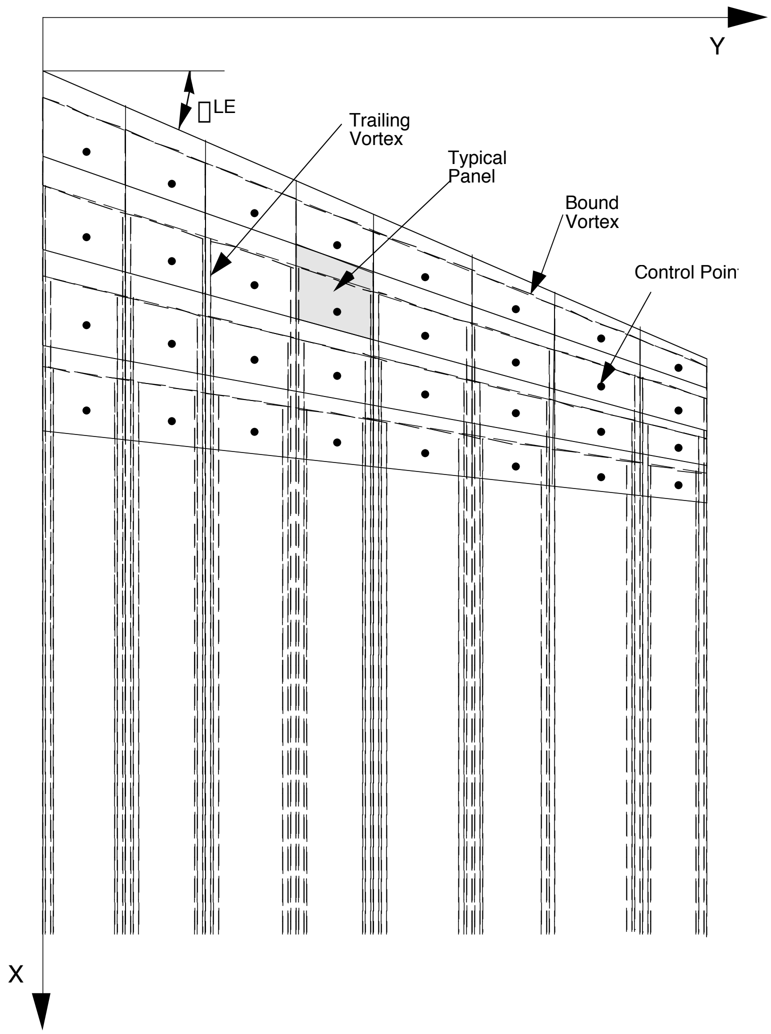

Figure 1.2: Elemental Panels for Lifting Surface Method (Ref. 3)

Like Classical VLM's, AeroLS has bound vortex segments that coincide with the quarter-chord line of the panel that are aligned with the local sweep angle. The vortex lattice panels are located on the mean-camber surface of the wing and the trailing vortices follow the curvature of the wing to the trailing edge before departing to downstream infinity.

AeroLS does not linearize the boundary conditions and vortex segments to the wing-chord plane as is done in most Classical VLM's, but rather, solves the boundary conditions on the mean-camber surface itself. Also, AeroLS does not make the small angle linearizing assumption often used in Classical VLM's (replacing sin(□) with // in radians and replacing cos(□) with *1*, etc.). This *does not* mean that AeroLS is valid at high angles of attack! Like all linear aero codes, AeroLS, is only valid in the "linear lift region" at low angles of attack and sideslip where the assumptions listed above are reasonably valid.

AeroLS assumes that vortices trailing downstream of a wing are straight, parallel to the X axis and are not affected by angle of attack, sideslip, or rotational rates. In reality this is not the case, but for most engineering applications, suitable accuracy is obtained by making this simplifying assumption.

2. Definition of the Panel Normal Vector and Panel Area:

The wing mean camber surface is represented by a system of quadrilateral panels. The method for determining each panel normal vector is straight forward. However, the method for finding the panel surface area is far more complicated than is really necessary for a lifting surface code. The reason for this is that the geometry library that AeroLS uses is very general and happens to implement the more formal method detailed here.

Since four points selected on a surface may not lie in the same plane, a mean surface through the four points is selected to represent the panel. The method for doing this was taken from a combination of Reference 1 and 2.

Let (x_i, y_i, z_i) represent the four points on the body surface.
Let (x'_i, y'_i, z'_i) represent the four points on the mean surface.

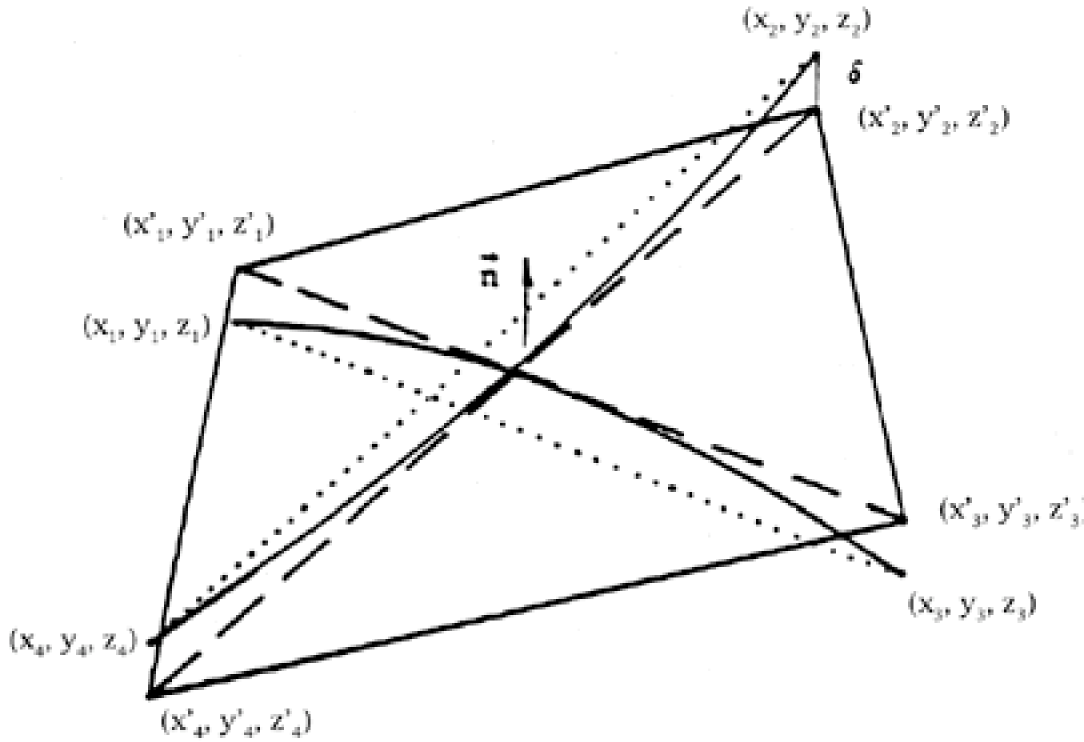

Figure 2.1: Representation of Mean Surface Through Four Points (Ref. 1)
(x,y,z) = corner points of actual panel, (x', y', z') = corner points of mean surface

The mean surface is chosen in the following manner (see Figure 2.1):

1. The direction of the panel normal is found by taking the cross product of the vectors representing the diagonals.

$$\hat{n} = \frac{\vec{d}_{31} \times \vec{d}_{42}}{\left|\vec{d}_{31} \times \vec{d}_{42}\right|} \tag{2.1}$$

Where:

$$\vec{d}_{31} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$
$$\vec{d}_{42} = (x_4 - x_2)\hat{i} + (y_4 - y_2)\hat{j} + (z_4 - z_2)\hat{k}$$

2. Given the normal vector, the plane of the element is completely determined if a point in this plane is specified. This point is taken as the point who's coordinates are the averages of the four original points

$$\bar{x} = \tfrac{1}{4}\left[x_1 + x_2 + x_3 + x_4\right] \qquad (2.2)$$
$$\bar{y} = \tfrac{1}{4}\left[y_1 + y_2 + y_3 + y_4\right]$$
$$\bar{z} = \tfrac{1}{4}\left[z_1 + z_2 + z_3 + z_4\right]$$

3. The original points are equidistant from the equivalent plane. The out of plane distance, δ, to these points is calculated as follows.

$$\delta = \left| n_x(\bar{x} - x_1) + n_y(\bar{y} - y_1) + n_z(\bar{z} - z_1)\right| \qquad (2.2)$$

4. The coordinates of the mean surface are calculated by adding or subtracting $\delta\hat{n}$ from each of the corner points.

$$x'_k = x_k + (-1)^{k+1}\, n_x\delta \qquad (2.3)$$
$$y'_k = y_k + (-1)^{k+1}\, n_y\delta$$
$$z'_k = z_k + (-1)^{k+1}\, n_z\delta$$

Where:

k = 1, 2, 3, or 4.

5. Now the element coordinate system must be constructed. This requires three mutually perpendicular unit vectors, one of which points along each of the element coordinate axes and it requires the coordinates of the origin of the coordinate system. The unit normal is taken as one of the unit vectors, so two perpendicular unit vectors in the plane of the mean surface element are needed. Denote these unit vectors as $\hat{t}_1$, and $\hat{t}_2$. The vector $\hat{t}_1$ is taken as $\vec{d}_{31}$ divided by it's own length.

$$\hat{t}_1 = \frac{\vec{d}_{31}}{\left|\vec{d}_{31}\right|} \qquad (2.4)$$

The vector $\hat{t}_2$ is defined by:

$$\hat{t}_2 = \hat{n} \times \hat{t}_1 \qquad (2.5)$$

The vector $\hat{t}_1$ is the unit vector parallel to the x or ξ axis of the element coordinate system, while $\hat{t}_2$ is parallel to the y or η axis, and $\hat{n}$ is parallel to the z or ζ axis of this coordinate system.

6. The corner points are now transformed into the element coordinate system based on the average point as the origin. These points have coordinates (x'_k, y'_k, z'_k) in the reference coordinate system. Their coordinates in the element coordinate system are denoted by $(\xi_k, \eta_k, 0)$. Because they lie in the plane of the element, they have zero z or ζ coordinate in the element coordinate system. Also, because the vector $\hat{t}_1$ is a multiple of the diagonal vector from point 1 to 3, the coordinate η_1 and η_3 are the equal. In the element coordinate system, the corner points are:

$$\xi_k = t_{1x}(\bar{x} - x'_k) + t_{1y}(\bar{y} - y'_k) + t_{1z}(\bar{z} - z'_k) \quad (2.6)$$

$$\eta_k = t_{2x}(\bar{x} - x'_k) + t_{2y}(\bar{y} - y'_k) + t_{2z}(\bar{z} - z'_k) \quad (2.7)$$

Where:

k = 1, 2, 3, or 4.

These corner points are taken as the corners of a plane quadrilateral as shown in Figure 2.2.

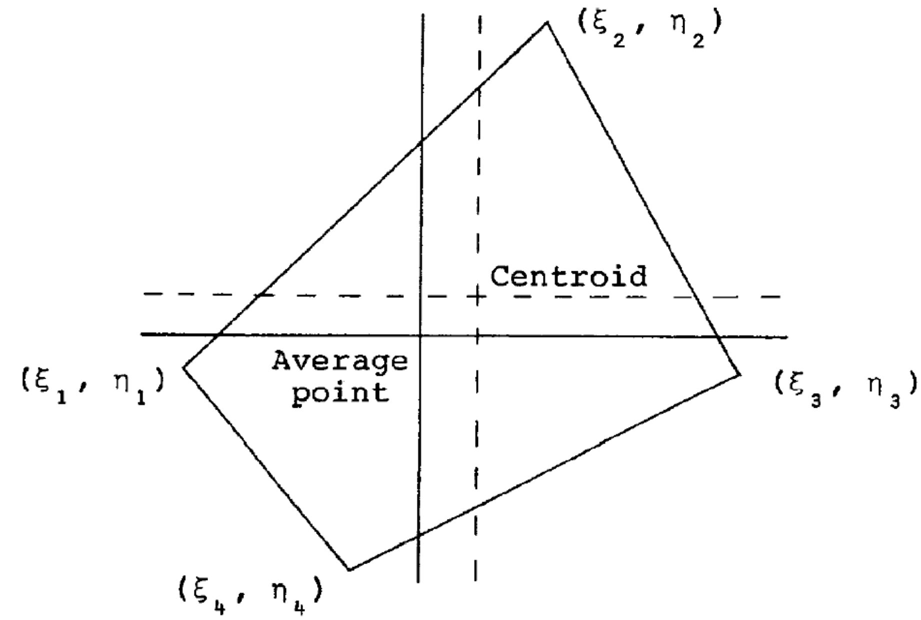

Figure 2.2: Mean Surface Element Corner Points (Ref. 2)

7. The origin of the element coordinate system is now transferred to the centroid of the area of the quadrilateral. With the average point as the origin, the coordinates of the centroid in the element coordinate system are:

$$\xi_0 = \frac{1}{3}\frac{1}{\eta_2 - \eta_4}\left[\xi_4(\eta_3 - \eta_2) + \xi_2(\eta_4 - \eta_1)\right] \quad (2.8)$$

$$\eta_0 = -\tfrac{1}{3}\eta_1 \quad (2.9)$$

These are subtracted from the coordinates of the corner points to obtain the coordinates of the corner points in an element coordinate system that has the centroid as the origin.

$$\square_k = \square_k \square \square_0 \tag{2.10}$$

$$\square_k = \square_k \square \square_0 \tag{2.11}$$

Where:

k = 1, 2, 3, or 4.

8. The centroid of the mean surface element in the reference coordinate system is given by:

$$x_0 = \overline{x} + t_{1x}\square_0 + t_{2x}\square_0 \tag{2.12}$$

$$y_0 = \overline{y} + t_{1y}\square_0 + t_{2y}\square_0$$

$$z_0 = \overline{z} + t_{1z}\square_0 + t_{2z}\square_0$$

9. Finally, the area of the mean surface element quadrilateral is:

$$A = \tfrac{1}{2}(\square_3 \square \square_1)(\square_2 \square \square_4) \tag{2.13}$$

3. Definition of Boundary Conditions:

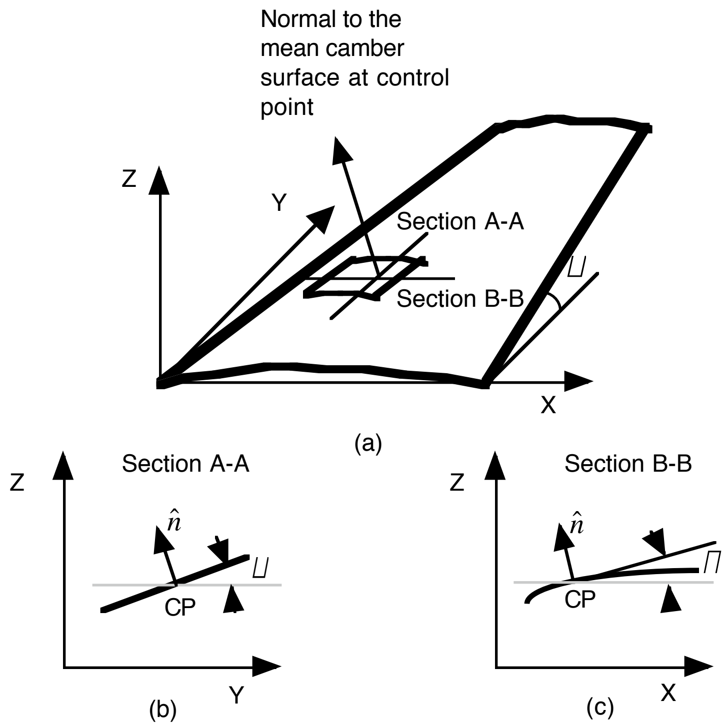

Figure 3.1: Geometry for the Tangency Boundary Condition (Ref. 3)

The boundary condition used in AeroLS is that the flow is tangent to the wing surface (Figure 3.1a). This means that the normal component of induced velocity due to the network of vortices at the control point of each panel must exactly balance the normal component of free stream velocity. In equation form, for each panel *m*, this is stated as:

$$\vec{V}_{TOT} \cdot \hat{n}_m = 0 \qquad (3.1)$$

where:

$$\vec{V}_{TOT} = \vec{V} + \vec{V}_{Im} \qquad (3.2)$$

where:

$\vec{V}_{Im}$ = The induced velocity at the control point of panel *m* due to the circulation of all the vortices in the system.

Rearranging:

$$-\vec{V}_{Im} \cdot \hat{n}_m = \vec{V} \cdot \hat{n}_m \qquad (3.3)$$

or

$$w_m = V_{Nm} \qquad (3.4)$$

where:

$V_{Nm} = \vec{V} \cdot \hat{n}_m$ = Magnitude of free stream velocity component normal to the panel control point.

$w_m = -\vec{V}_{Im} \cdot \hat{n}_m$ = Downwash at the control point of panel *m*.

The free stream velocity can be written in component form as:

$$\vec{V} = U\hat{i} + V\hat{j} + W\hat{k} \tag{3.5}$$

or

$$\vec{V} = V\ (\cos\square\cos\square\hat{i}\ \square\cos\square\sin\square\hat{j} + \sin\square\hat{k}) \tag{3.6}$$

Dotting Equation 3.6 with the panel normal vector gives the normal component of free stream velocity at the panel control point:

$$V_{Nm} = V\ (\cos\square\cos\square\cdot n_x\ \square\cos\square\sin\square\cdot n_y + \sin\square\cdot n_z) \tag{3.7}$$

Substituting Equations 3.7 into Equation 3.4 we get:

$$w_m = V\ (\cos\square\cos\square\cdot n_x\ \square\cos\square\sin\square\cdot n_y + \sin\square\cdot n_z) \tag{3.8}$$

Equation 3.8 only applies to the m^{th} panel. If we use matrix notation, we can write a similar equation that accounts for all panels.

$$\{w\} = V\ \{\square_{N0}\} \tag{3.9}$$

where:

$\square_{N0}$ = "angle of attack" normal to the plane of the panel.

$$\{\square_{N0}\} = \{\cos\square\cos\square\cdot n_x\ \square\cos\square\sin\square\cdot n_y + \sin\square\cdot n_z\} \tag{3.10}$$

Equation 3.9 forms the basic boundary condition for determining vortex strengths in Section 4. Equation 3.10 will prove useful when determining stability derivatives.

4. Velocity Induced by a Horseshoe Vortex:

The velocity induced by a vortex filament of strength Γ and length dl (see Figure 4.1) is given by the law of Biot and Savart (Reference 3):

$$d\vec{V} = \frac{\Gamma\left(d\vec{l} \times \vec{r}\right)}{4\pi r^3} \tag{4.1}$$

The magnitude is:

$$dV = \frac{\Gamma \sin\theta \, dl}{4\pi r^2} \tag{4.2}$$

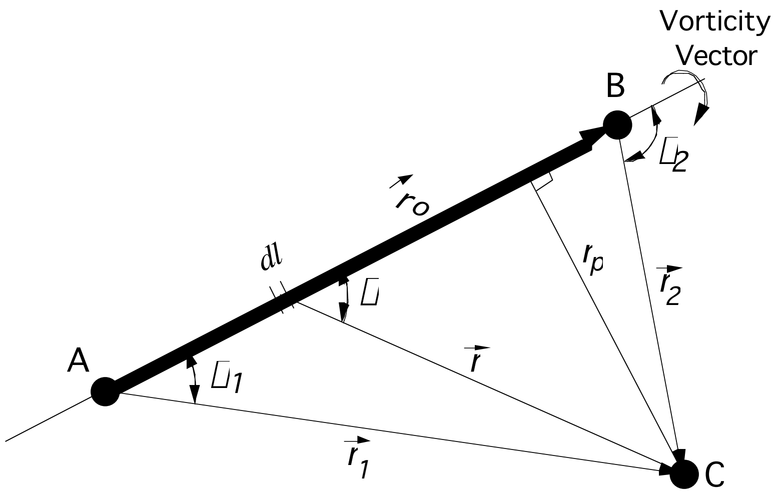

Figure 4.1: Vortex Segment Geometry

We can integrate between A and B in Figure 4.1 to get the magnitude of the induced velocity due to a vortex segment (Reference 3):

$$V = \frac{\Gamma}{4\pi r_p}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = \frac{\Gamma}{4\pi r_p}\left(\cos\theta_1 - \cos\theta_2\right) \tag{4.3}$$

Let $\vec{r}_0$, $\vec{r}_1$ and $\vec{r}_2$ designate the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ respectively as shown in Figure 4.1. Then (from Reference 3):

$$r_p = \frac{\left|\vec{r}_1 \times \vec{r}_2\right|}{r_o} \qquad \cos\theta_1 = \frac{\vec{r}_0 \cdot \vec{r}_1}{r_o r_1} \qquad \cos\theta_2 = \frac{\vec{r}_0 \cdot \vec{r}_2}{r_o r_2} \tag{4.4a, b, c}$$

$$\textit{direction of induced velocity} = \frac{\vec{r}_1 \times \vec{r}_2}{\left|\vec{r}_1 \times \vec{r}_2\right|} \tag{4.4d}$$

Substituting these into Equation 4.3 yields:

$$V = \frac{\Gamma_n}{4\pi}\frac{\vec{r}_1 \times \vec{r}_2}{\left|\vec{r}_1 \times \vec{r}_2\right|^2}\left[\vec{r}_0 \cdot \left(\frac{\vec{r}_1}{r_1} - \frac{\vec{r}_2}{r_2}\right)\right] = \vec{C}\,\Gamma_n \qquad (4.4)$$

This is the basic equation giving the induced velocity at a point in space due to a single vortex segment of finite length. It can be used regardless of the orientation of the vortex.

We will now use Equation 4.4 to calculate the velocity that is induced at a point in space due to a horseshoe vortex as shown in Figure 4.2. Figure 4.2 shows a set of quadrilateral panels that represent the mean camber surface of a wing. The horseshoe vortex can be assumed to represent the vorticity induced by a single wing panel (e.g.: the n^{th} panel). The segment *AB* in Figure 4.4 represents the bound vortex and coincides with the 1/4 chord line of the panel *n*. The trailing vortices start at the inboard and outboard edge of the 1/4 chord line of the panel and follow the contour of the surface aft to the trailing edge of the lifting surface, and then extend infinitely far aft in a straight line.

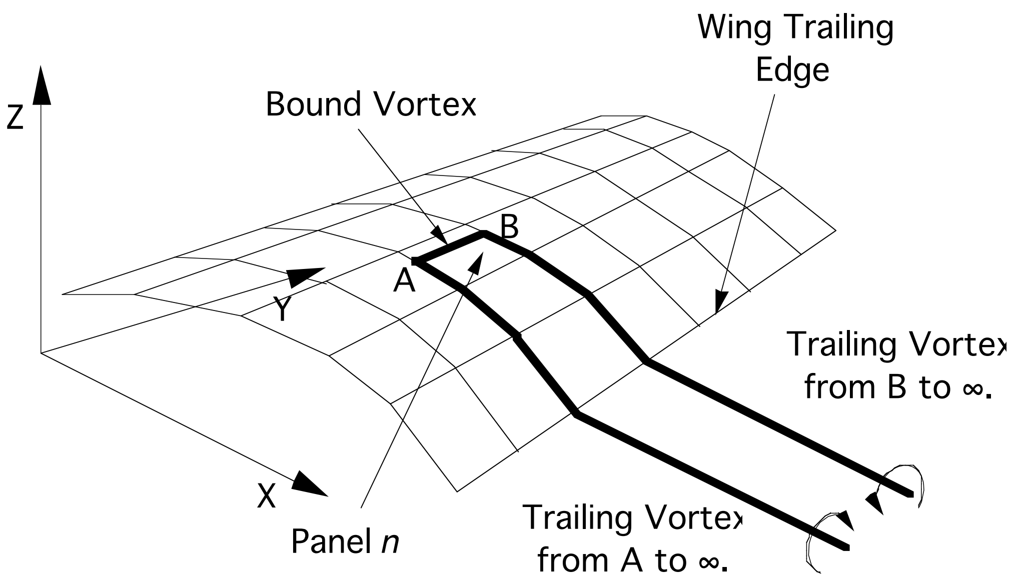

Figure 4.2: A Conformal Horseshoe Vortex as Used in AeroLS

The induced velocity at a point in space due to the horseshoe vortex will be calculated by adding together the influence of each straight line segment of the horseshoe calculated using Equation 4.4 (see Figure 4.3).

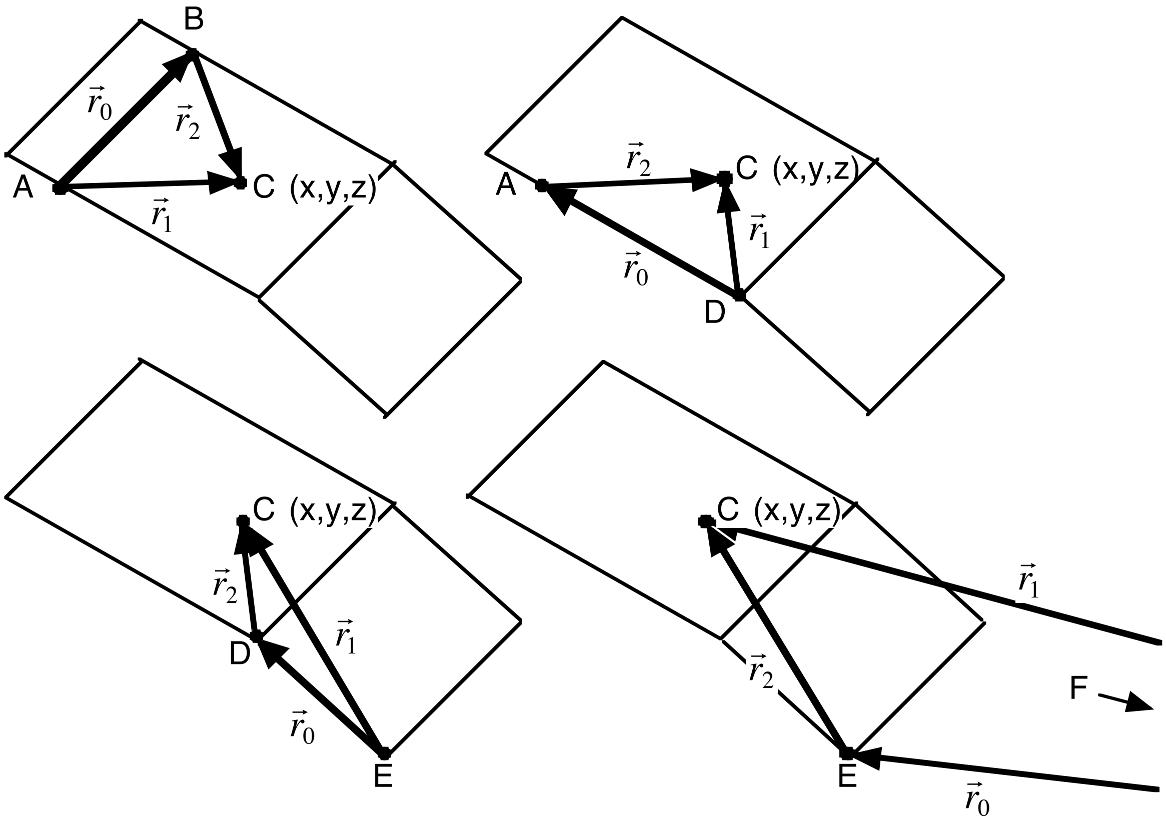

Figure 4.3: Vector Elements for Calculation of Induced Velocity

For the bound vortex segment $\overline{AB}$, the vectors used in Equation 4.4 are defined as:

$$\vec{r}_0 = \overline{AB} = (xB_n - xA_n)\hat{i} + (yB_n - yA_n)\hat{j} + (zB_n - zA_n)\hat{k} \qquad (4.5a)$$

$$\vec{r}_1 = (x - xA_n)\hat{i} + (y - yA_n)\hat{j} + (z - zA_n)\hat{k} \qquad (4.5b)$$

$$\vec{r}_1 = (x - xB_n)\hat{i} + (y - yB_n)\hat{j} + (z - zB_n)\hat{k} \qquad (4.5c)$$

For the panel side edge vortex segment, from the panel trailing edge to the panel 1/4 chord, $\overline{DA}$, the vectors used in Equation 4.4 are defined as:

$$\vec{r}_0 = \overline{DA} = (xA_n - xD_n)\hat{i} + (yA_n - yD_n)\hat{j} + (zA_n - zD_n)\hat{k} \qquad (4.6a)$$

$$\vec{r}_1 = (x - xD_n)\hat{i} + (y - yD_n)\hat{j} + (z - zD_n)\hat{k} \qquad (4.6b)$$

$$\vec{r}_1 = (x - xA_n)\hat{i} + (y - yA_n)\hat{j} + (z - zA_n)\hat{k} \qquad (4.6c)$$

Next we need to step from one panel to another until we reach the trailing edge of the lifting surface. In the example shown in Figure 4.3, there is only one panel between the panel being evaluated and the trailing edge. For the next aft panel side edge vortex segment, from the panel trailing edge to the panel leading edge, $\overline{ED}$, the vectors used in Equation 4.4 are defined as:

$$\vec{r}_0 = \overline{ED} = (xD_n - xE_n)\hat{i} + (yD_n - yE_n)\hat{j} + (zD_n - zE_n)\hat{k} \qquad (4.7a)$$

$$\vec{r}_1 = (x - xE_n)\hat{i} + (y - yE_n)\hat{j} + (z - zE_n)\hat{k} \qquad (4.7b)$$

$$\vec{r}_1 = (x - xD_n)\hat{i} + (y - yD_n)\hat{j} + (z - zD_n)\hat{k} \qquad (4.7c)$$

If there were more than one panel between the n^{th} panel and the lifting surface trailing edge, Equation 4.4 would have to be evaluated for each segment.

For the filament that extends from the lifting surface trailing edge, E in Figure 4.3, to ∞, Reference 4 shows that it is sufficiently accurate to approximate the infinite length vortex filament using a finite length segment that extends from E to F and then use Equation 4.4 directly. In this case, point F is placed an arbitrary, but large distance aft (AeroLS uses 100 times the largest Y coordinate in the input geometry). For the vortex segment trailing from infinity to the lifting surface trailing edge, $\overline{FE}$, the vectors used in Equation 4.4 are defined as:

$$\vec{r}_0 = \overline{FE} = (xE_n - xF_n)\hat{i} + (yE_n - yF_n)\hat{j} + (zE_n - zF_n)\hat{k} \qquad (4.8a)$$

$$\vec{r}_1 = (x - xF_n)\hat{i} + (y - yF_n)\hat{j} + (z - zF_n)\hat{k} \qquad (4.8b)$$

$$\vec{r}_1 = (x - xE_n)\hat{i} + (y - yE_n)\hat{j} + (z - zE_n)\hat{k} \qquad (4.8c)$$

The calculations, using Equation 4.4, for each of these segments is then repeated for the trailing vortices on the other, outboard, side of the n^{th} panel and the outboard side of all the panels downstream of the n^{th} panel.

The total velocity induced at some point (x, y, z) by the horseshoe vortex representing a surface element (e.g.: for the n^{th} panel) is the sum of the components: bound plus each side edge of panel n plus each side edge of all downstream panels plus each trailing edge to ∞. Let the point (x, y, z) be the control point of the m^{th} panel with coordinates (x_m, y_m, z_m). The velocity induced at the m^{th} control point by the vortex representing the n^{th} panel will be:

$$\vec{V}_{I\,m,n} = \vec{C}_{m,n}\Gamma_n \qquad (4.9)$$

Where:

$\vec{C}_{m,n}\Gamma_n$ = The induced velocity at the control point of panel m induced by all the vortex segments associated with panel n.

The component of Equation 4.9 normal to the panel containing the control point is:

$$\vec{V}_{I\,m,n} \cdot \hat{n}_m = \left(\vec{C}_{m,n} \cdot \hat{n}_m\right)\Gamma_n \qquad (4.10)$$

Now we introduce the downwash parameter, w_m, as we did in Section 3:

$$w_m = -\vec{V}_{I\,m} \cdot \hat{n}_m = -\sum_{n=1}^{n=2N} \vec{V}_{I\,m,n} \cdot \hat{n}_m \qquad (4.11)$$

and:

$$W_{m,n} = -\vec{C}_{m,n} \cdot \hat{n}_m \qquad (4.12)$$

Where:

$W_{m,n}$ is the downwash on panel m due to a unit circulation on panel n.

We can put Equations 4.10, 4.11 and 4.12 together into matrix notation for all panels as follows:

$$\{w\} = [W]\{□\} \tag{4.13}$$

Equation 4.13 reads: the downwash, $\{w\}$, on each panel is equal to the downwash matrix, $[W]$, times the vortex circulation strengths of each panel, $\{□\}$.

The downwash matrix can be calculated from Equation 4.12 by setting the circulation strengths (□) in Equation 4.4 to 1.0. For every panel, *m*, sum up the induced velocity of unit circulation strengths associated with all the vortex segments of another panel, *n*. The component (dot product) of that induced velocity normal to the panel is the downwash on panel *m* due to a unit circulation strength associated with panel *n*. The downwash matrix is a function of geometry alone.

$\{w\}$ is calculated from the boundary condition of Equation 3.9 and is a function of the flight condition (angle of attack, angle of sideslip) and possibly flap deflection via modification of the panel normal vectors.

With the downwash vector and the downwash matrix known, it is now possible to solve for the vortex circulation strengths, $\{□\}$, that make Equation 4.13 true. This could be done using any standard linear algebra library and involves calculating the inverse of the downwash matrix. However, it is more computationally efficient to hold off solving the system of equations until you are ready to find the loads on the panels in the next section.

5. Calculation of Loads and Aerodynamic Coefficients:

The component of load normal to each panel can be written as (Reference 5):

$$\{L_N\} = \rho V \, [b]\{\Gamma\} \tag{5.1}$$

where:

$\{L_N\}$ = The component of load normal to each panel's surface

ρ = The air mass density

V = The free stream velocity magnitude

$[b]$ = A diagonal vector containing the true (not projected) span of each panel.

$\{\Gamma\}$ = Vortex circulation strength associated with each panel.

By combining Equations 4.13 and 3.9 and rearranging, we can see that

$$\{\Gamma\} = V \, [W]^{-1}\{\alpha_{N0}\} \tag{5.2}$$

If we substitute Equation 5.2 into Equation 5.1 we get:

$$[W]\{L_N\} = 2\bar{q}[b]\{\alpha_{N0}\} \tag{5.3}$$

where:

$\bar{q} = \frac{1}{2}\rho V^2$ = Dynamic pressure

If we cast Equation 5.3 in terms of a "panel normal load coefficient", by dividing through by the reference area and the dynamic pressure, we get:

$$[W]\{C_{L_N}\} = \frac{2}{S_{\mathrm{Re}f}}[b]\{\alpha_{N0}\} \tag{5.4}$$

where:

$\{C_{L_N}\}$ = The load coefficient normal to each panel's surface.

$S_{\mathrm{Re}f}$ = The reference wing area

Equation 5.4 is a system of linear equations where the unknowns are the vector of load coefficients, $\{C_{L_N}\}$, one for each panel. Equation 5.4 is efficiently solved using a linear algebra system of equations solver.

Once the normal load coefficient at each panel has been computed, the aerodynamic coefficients can be computed as follows:

The force normal to the aircraft body X-axis (F_Z) is the sum of the forces normal to the surface of each panel times the Z component of the normal vector of each panel.

$$F_z = \sum_{2N} \{L_N\}\{n_z\} \tag{5.5}$$

Where:

$\{n_z\}$ = Z component of the normal vector of each panel.

In terms of normal force coefficient (C_N or C_Z – force coefficient in the positive Z or upward direction):

$$C_N = C_z = \sum_{n=1}^{2N} \{C_{L_N}\}\{n_z\} = \frac{2}{S_{\mathrm{Ref}}} \sum_{n=1}^{2N} [W]^{-1}[b]\{\alpha_N\}\{n_z\} \tag{5.6}$$

If symmetry is assumed, and a symmetric downwash matrix is computed, then:

$$C_N = 2\sum_{n=1}^{N} \{C_{L_N}\}\{n_z\} \tag{5.7}$$

Since this method is inviscid, and the angles of camber/slope tend to be small, axial force, F_X, is usually near zero. However, AeroLS calculates it anyway using the following equations:

$$F_x = \sum_{2N} \{L_N\}\{n_x\} \tag{5.8}$$

Where:

$\{n_x\}$ = X component of the normal vector of each panel.

In terms of body axial force coefficient (C_A or C_X – force coefficient in the positive X or aft direction):

$$C_A = C_x = \sum_{n=1}^{2N} \{C_{L_N}\}\{n_x\} \tag{5.9}$$

If symmetry is assumed, and a symmetric downwash matrix is computed, then:

$$C_A = 2\sum_{n=1}^{N} \{C_{L_N}\}\{n_x\} \tag{5.10}$$

The body axis side force, F_Y, is the sum of the forces normal to the surface of each panel times the Y component of the normal vector of each panel:

$$F_y = \sum_{2N} \{L_N\}\{n_y\} \tag{5.11}$$

Where:

$\{n_y\}$ = Y component of the normal vector of each panel.

In terms of body side force coefficient (C_{Yb} or C_y – force coefficient in the positive Y or starboard direction):

$$C_{Yb} = C_y = \sum_{n=1}^{2N} \{C_{L_N}\}\{n_y\} \tag{5.12}$$

If anti-symmetry is assumed, and an anti-symmetric downwash matrix is used to calculate the normal load coefficient on each panel, then:

$$C_{Yb} = 2\sum_{n=1}^{N} \{C_{L_N}\}\{n_y\} \tag{5.13}$$

With the body axis forces (F_x, F_y, & F_z, or C_N, C_A, and C_{Yb}) known, it is possible to resolve them into the wind axis force coefficients as follows:

$$C_L = C_N \cos\alpha - C_A \sin\alpha \tag{5.14}$$

$$C_D = C_N \sin\alpha\cos\beta + C_A \cos\alpha\cos\beta - C_{Yb} \sin\beta \tag{5.15}$$

$$C_Y = C_N \sin\alpha \sin\beta + C_A \cos\alpha\sin\beta + C_{Yb} \cos\beta \tag{5.16}$$

These can be further resolved into stability axis coefficients using the following equations:

$$C_{L_{SA}} = C_L \tag{5.17}$$

$$C_{D_{SA}} = C_D \cos\beta - C_Y \sin\beta \tag{5.18}$$

$$C_{Y_{SA}} = C_Y \cos\beta + C_D \sin\beta \tag{5.19}$$

The body-axis pitching moment is the sum of the normal forces (in the Z direction) on each panel times the moment arm of each panel in the X direction:

$$M = -\sum_{2N} [X - X_{C.G.}]\{F_z\} \tag{5.20}$$

Where:

M = Pitching moment about body Y axis (positive nose up).
X = The X location of the 1/4 mean geometric chord of each panel.
$X_{C.G.}$ = The X location of the reference point or center of gravity.

or, in coefficient form, assuming symmetry:

$$C_{Mb} = -\frac{2}{c_{\mathrm{Ref}}}\sum_{n=1}^{N} [n_z(X - X_{C.G.})]\{C_{L_N}\} \tag{5.21}$$

The body-axis rolling moment (positive starboard wing down) is the sum of the panel forces in the body Y direction times the Z moment arm minus the sum of the panel forces in the Z direction times the Y moment arm:

$$\mathcal{L} = [Z - Z_{C.G.}]\{F_y\} - [Y - Y_{C.G.}]\{F_z\} \tag{5.22}$$

or

$$\mathcal{L} = \left([Z - Z_{C.G.}]\{n_y\} - [Y - Y_{C.G.}]\{n_z\}\right)\{L_N\} \tag{5.23}$$

Where:

$\mathcal{L}$ = Rolling moment about body X-axis (positive right wing down).
Z = The Z location of the 1/4 mean geometric chord of each panel.
$Z_{C.G.}$ = The Z location of the reference point or center of gravity.
Y = The Y location of the 1/4 mean geometric chord of each panel.
$Y_{C.G.}$ = The Y location of the reference point or center of gravity.

or, in coefficient form, assuming anti-symmetry:

$$C_{lb} = 2\sum_{n=1}^{N}\left[n_y(Z - Z_{C.G.}) - n_z(Y - Y_{C.G.})\right]\{C_{L_N}\} \tag{5.24}$$

Finally, the body-axis yawing moment (positive starboard wing aft) is the sum of the panel forces in the body Y axis direction times the X moment arm:

$$N = -\sum_{2N}[X - X_{C.G.}]\{F_y\} \tag{5.25}$$

Where:

N = Yawing moment about body Z-axis (positive right wing aft).

or, in coefficient form, assuming anti-symmetry:

$$C_{nb} = -2\sum_{n=1}^{N}\left[n_y(X - X_{C.G.})\right]\{C_{L_N}\} \tag{5.26}$$

With the body axis moments (C_{Mb}, C_{lb}, and C_{nb}) known, it is possible to resolve them into the wind axis moment coefficients as follows:

$$C_M = C_{Mb}\cos\beta - C_{lb}\frac{b_{ref}}{c_{ref}}\cos\alpha\sin\beta - C_{nb}\frac{b_{ref}}{c_{ref}}\sin\alpha\sin\beta \tag{5.27}$$

$$C_l = C_{Mb}\frac{c_{ref}}{b_{ref}}\sin\beta + C_{lb}\cos\alpha\cos\beta + C_{nb}\sin\alpha\cos\beta \tag{5.28}$$

$$C_n = C_{nb}\cos\alpha - C_{lb}\sin\alpha \tag{5.29}$$

These can be further resolved into stability axis coefficients using the following equations:

$$C_{M_{SA}} = C_M\cos\beta - C_l\frac{b_{ref}}{c_{ref}}\sin\beta \tag{5.30}$$

$$C_{l_{SA}} = C_M\frac{c_{ref}}{b_{ref}}\sin\beta + C_l\cos\beta \tag{5.31}$$

$$C_{n_{SA}} = C_n \tag{5.32}$$

6. Calculation of α and β Stability Derivatives:

Many stability derivatives can be found analytically from the underlying equations that make up the vortex lattice method, and so do not have to be found by differencing as is done in other types of analysis.

It is of interest to find the lift curve slope, $C_{L_\alpha} = C_{N_\alpha}\Big|_{\beta=0}$, and the body axis pitching moment curve slope, C_{Mb_a}. Let's derive the calculation of normal force slope as an example:

$$C_{N_a} = \frac{\partial C_N}{\partial \alpha} \tag{6.1}$$

From Equation 5.7, it can be seen that:

$$\frac{\partial C_N}{\partial \alpha} = 2\sum_{n=1}^{N} \left\{\frac{\partial C_{L_N}}{\partial \alpha}\right\}\{n_z\} \tag{6.2}$$

From Equation 5.4, it can be seen that:

$$[W]\left\{\frac{\partial C_{L_N}}{\partial \alpha}\right\} = \frac{2}{S_{\mathrm{Re}f}}[b]\left\{\frac{\partial \Lambda_{N0}}{\partial \alpha}\right\} \tag{6.3}$$

And finally, from Equation 3.10 it can be seen that:

$$\left\{\frac{\partial \Lambda_{N0}}{\partial \alpha}\right\} = \left\{-\sin\alpha\cos\beta \cdot n_x + \sin\alpha\sin\beta \cdot n_y + \cos\alpha \cdot n_z\right\} \tag{6.4}$$

Notice that Equations 6.2, 6.3, and 6.4 are identical in form to the original equations used to calculate the normal force coefficient in Section 5. The solution process is identical. In fact, the exact same algorithms and subroutines can be used to calculate normal force coefficient or normal force coefficient slope by simply substituting equation 6.4 for equation 3.10 in the original formulation. This is the approach used by AeroLS.

Just as in the original formulation, once you have solved for the value of $\frac{\partial C_{L_N}}{\partial \alpha}$ at every panel, all the alpha stability derivatives can be calculated using the same summations used for the forces & moments in Section 5.

The same process can be repeated to calculate C_{Mb_β}.

Likewise, for the sideslip derivatives, using side force as an example:

$$C_{Yb_{\beta}} = \frac{\partial C_{Yb}}{\partial \beta} \tag{6.5}$$

From Equation 5.13, it can be seen that:

$$\frac{\partial C_{Yb}}{\partial \beta} = 2\sum_{n=1}^{N} \left\{ \frac{\partial C_{L_N}}{\partial \beta} \right\} \left\{ n_y \right\} \tag{6.6}$$

From Equation 5.4, it can be seen that:

$$\left[W\right] \left\{ \frac{\partial C_{L_N}}{\partial \beta} \right\} = \frac{2}{S_{\mathrm{Re}f}} \left[b\right] \left\{ \frac{\partial \alpha_{N0}}{\partial \beta} \right\} \tag{6.7}$$

And finally, from Equation 3.10 it can be seen that:

$$\left\{ \frac{\partial \alpha_{N0}}{\partial \beta} \right\} = \left\{ -\cos\alpha \sin\beta \cdot n_x - \cos\alpha \cos\beta \cdot n_y + 0 \cdot n_z \right\} \tag{6.8}$$

Just as with the alpha derivatives, the exact same process is used to calculate the sideslip derivatives as the lateral-directional forces and moments. The only change is substituting Equation 6.8 for Equation 3.10.

This same process can be repeated for $C_{nb_{\beta}}$ and $C_{lb_{\beta}}$.

This same technique can be used to find the control derivatives associated with a flap or control surface deflection by taking the derivative of Equation 3.10 with respect to the control surface deflection of interest and then solving for all the panel load slopes and control derivatives in exactly the same manner as outlined above. In the case of a flap or control surface deflection, the slopes would be in the normal vector components. AeroLS does not yet calculate control surface derivatives.

7. Calculation of Rotary Stability Derivatives:

7.1 Pitch Damping

(Taken from Reference 5)

Pitch damping derivatives are based on the pitch rate angle, $\hat{q}$:

$$\hat{q} = \frac{Q \cdot c_{\mathrm{Ref}}}{2V} \tag{7.1}$$

Where:

Q = Aircraft pitch rate in radians/second.
V = Free stream true air speed (units consistent with c_{Ref} and Q).

Consider an airplane at constant speed, angle of attack, and pitch rate. It must be flying at constant radius, r, about a point and:

$$Q = \frac{V}{r} \tag{7.2}$$

Lay out a length $c_{\mathrm{Ref}}/2$ along the X-axis, then the pitch rate angle is defined as shown in Figure 7.1.

$$r = \frac{V}{Q} \qquad \frac{c_{\mathrm{Ref}}}{2r} = \frac{Qc_{\mathrm{Ref}}}{2V} = \hat{q}$$

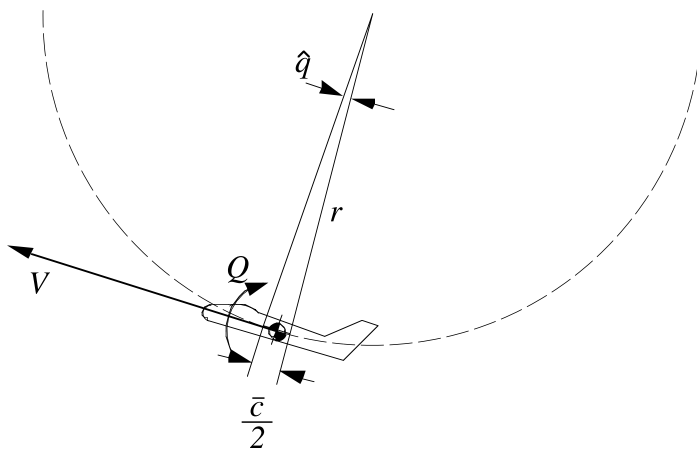

Figure 7.1: Definition of Pitch Rate Angle (Ref. 5)

The airplane "sees" an apparent curvature of the airflow. The local flow angle varies along the length of the airplane. Flow angle due to pitch rate is then:

$$\{\alpha_Q\} = \{X - X_{C.G.}\}\frac{Q}{V} \tag{7.3}$$

Where:

$\{\alpha_Q\}$ = The component of angle of attack at each panel due to pitch rate.

The component of angle of attack, normal to each panel, due to pitch rate is then:

$$\left\{\alpha_{Q_N}\right\} = \left[X - X_{C.G.}\right]\left\{n_z\right\}\frac{Q}{V} \tag{7.4}$$

or

$$\left\{\alpha_{Q_N}\right\} = \frac{2}{c_{\mathrm{Re}f}}\left[X - X_{C.G.}\right]\left\{n_z\right\}\frac{Qc_{\mathrm{Re}f}}{2V} \tag{7.5}$$

or

$$\left\{\alpha_{Q_N}\right\} = \frac{2}{c_{\mathrm{Re}f}}\left[X - X_{C.G.}\right]\left\{n_z\right\}\cdot\hat{q} \tag{7.6}$$

This can be treated just like a camber vector. Think of it as curving the airplane rather than the airflow. Equation 7.6 can be added to Equation 3.10 to get:

$$\left\{\alpha_N\right\} = \left\{\alpha_{N_0}\right\} + \left\{\alpha_{Q_N}\right\} \tag{7.7}$$

By substituting this equation for $\left\{\alpha_N\right\}$ in place of $\left\{\alpha_{N_0}\right\}$ in Equation 3.9, the effect of a constant pitch rate can be taken into account in computing forces and moments.

It is of interest to compute the change in lift with a change in pitch rate angle, $C_{L_{\hat{q}}}$:

$$C_{L_{\hat{q}}} = C_{N_{\hat{q}}}\Big|_{\alpha=0} = \frac{\partial C_N}{\partial\hat{q}} \tag{7.8}$$

From Equation 5.7, it can be seen that:

$$\frac{\partial C_N}{\partial\hat{q}} = 2\sum_{n=1}^{N}\left\{\frac{\partial C_{L_N}}{\partial\hat{q}}\right\}\left\{n_z\right\} \tag{7.9}$$

From Equation 5.4, it can be seen that:

$$\left[W\right]\left\{\frac{\partial C_{L_N}}{\partial\hat{q}}\right\} = \frac{2}{S_{\mathrm{Re}f}}\left[b\right]\left\{\frac{\partial\alpha_N}{\partial\hat{q}}\right\} \tag{7.10}$$

Finally, from Equation 7.6 and 7.7, it can be seen that:

$$\left\{\frac{\partial\alpha_N}{\partial\hat{q}}\right\} = \left\{\frac{\partial\alpha_{Q_N}}{\partial\hat{q}}\right\} = \frac{2}{c_{\mathrm{Re}f}}\left[X - X_{C.G.}\right]\left\{n_z\right\} \tag{7.11}$$

Again, as with the angle of attack stability derivatives, the pitch damping derivatives can be calculated using exactly the same process as the longitudinal forces and moments, but with the substitution of Equation 7.11 for Equation 3.10.

The same process is used to compute $C_{Mb_{\hat{q}}}$.

7.2 Yaw Damping

(Taken from Reference 5)

The yaw damping phenomena is similar to the pitch damping except that the point, about which the airplane is flying, is the in the X-Y plane instead of the X-Z plane and the reference length is the reference span rather than the chord. Yaw rate angle, $\hat{r}$ is:

$$\hat{r} = \frac{R \cdot b_{\mathrm{Re}f}}{2V} \tag{7.12}$$

Where:

R = Aircraft yaw rate in radians/second.
V = Free stream true air speed (units consistent with $b_{\mathrm{Re}f}$ and R).

Consider an airplane at constant speed, angle of sideslip, and yaw rate. It must be flying at constant radius, r, about a point and:

$$R = \frac{V}{r} \tag{7.13}$$

Lay out a length $b_{\mathrm{Re}f}/2$ along the X-axis, then the yaw rate angle is defined as shown in Figure 7.2.

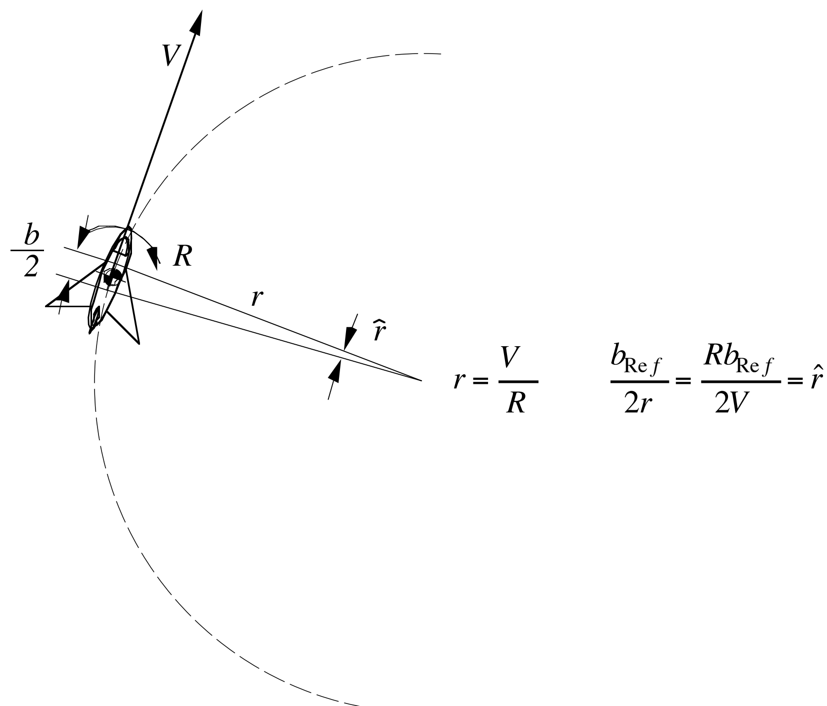

Figure 7.2: Definition of Yaw Rate Angle (Ref. 5)

The airplane "sees" an apparent curvature of the airflow. The local flow angle varies along the length of the airplane. Flow angle, or sidewash, due to yaw rate is then:

$$\{\beta_R\} = \pm\left[\sqrt{(X - X_{C.G.})^2 + (Y - Y_{C.G.})^2}\right]\frac{R}{V} \tag{7.14}$$

Where:

$\{\beta_R\}$ = The component of sideslip at each panel due to yaw rate.

The component of sidewash, normal to each panel, due to yaw rate is then:

$$\left\{\alpha_{R_N}\right\} = -\left(\left[X - X_{C.G.}\right]\left\{n_y\right\} + \left[Y - Y_{C.G.}\right]\left\{n_x\right\}\right)\frac{R}{V} \tag{7.15}$$

or

$$\left\{\alpha_{R_N}\right\} = \frac{-2}{b_{\mathrm{Re}f}}\left(\left[X - X_{C.G.}\right]\left\{n_y\right\} + \left[Y - Y_{C.G.}\right]\left\{n_x\right\}\right)\frac{R\,b_{\mathrm{Re}f}}{2V} \tag{7.16}$$

or

$$\left\{\alpha_{R_N}\right\} = \frac{-2}{b_{\mathrm{Re}f}}\left(\left[X - X_{C.G.}\right]\left\{n_y\right\} + \left[Y - Y_{C.G.}\right]\left\{n_x\right\}\right)\cdot \hat{r} \tag{7.17}$$

This can be treated just like a camber vector. Just as in pitch damping, think of it as curving the airplane rather than the airflow. Equation 7.17 can be added to Equation 3.10 to get:

$$\left\{\alpha_N\right\} = \left\{\alpha_{N_0}\right\} + \left\{\alpha_{R_N}\right\} \tag{7.18}$$

By substituting this equation for $\left\{\alpha_N\right\}$ in place of $\left\{\alpha_{N_0}\right\}$ in Equation 3.9, the effect of a constant yaw rate can be taken into account in computing forces and moments.

It is of interest to compute the change in side force with a change in yaw rate angle, $C_{Y_{\hat{r}}}$:

$$C_{Y_{\hat{r}}} = C_{Y_{\hat{r}}}\Big|_{\beta=0} = \frac{\partial C_Y}{\partial \hat{r}} \tag{7.19}$$

From Equation 5.13, it can be seen that:

$$\frac{\partial C_Y}{\partial \hat{r}} = 2\sum_{n=1}^{N}\left(\frac{\partial C_{L_N}}{\partial \hat{r}}\right)\left\{n_y\right\} \tag{7.20}$$

From Equation 5.4, it can be seen that:

$$\left[W\right]\left\{\frac{\partial C_{L_N}}{\partial \hat{r}}\right\} = \frac{2}{S_{\mathrm{Re}f}}\left[b\right]\left\{\frac{\partial \alpha_N}{\partial \hat{r}}\right\} \tag{7.21}$$

Finally, from Equation 7.17 and 7.18, it can be seen that:

$$\left\{\frac{\partial \alpha_N}{\partial \hat{r}}\right\} = \left\{\frac{\partial \alpha_{R_N}}{\partial \hat{r}}\right\} = \frac{-2}{b_{\mathrm{Re}f}}\left(\left[X - X_{C.G.}\right]\left\{n_y\right\} + \left[Y - Y_{C.G.}\right]\left\{n_x\right\}\right) \tag{7.22}$$

Again, as with the pitch damping derivatives, the yaw damping derivatives can be calculated using exactly the same process as the longitudinal forces and moments, but with the substitution of Equation 7.22 for Equation 3.10.

The same process is used to compute $C_{Nb_{\hat{r}}}$ and $C_{lb_{\hat{r}}}$.

7.3 Roll Damping

(Taken from Reference 5)

The roll damping derivatives are computed with respect to the wing tip helix angle, $\hat{p}$:

$$\hat{p} = \frac{P \cdot b_{\mathrm{Re}f}}{2V} \tag{7.23}$$

Where:

P = Aircraft roll rate (about the -X axis) in radians/second.

V = Free stream true air speed (units consistent with $b_{\mathrm{Re}f}$ and P).

Roll rate adds a component of angle of attack and sideslip over the surface of the airplane. The airplane "sees" an apparent curvature of the airflow. The local flow angle varies along the length and height of the airplane. Flow angle, or alpha, due to roll rate is then:

$$\{\alpha_P\} = \left(\sqrt{(Y - Y_{C.G.})^2 + (Z - Z_{C.G.})^2}\right)\frac{P}{V} \tag{7.24}$$

Where:

$\{\alpha_P\}$ = The component of alpha & beta at each panel due to roll rate.

The component of flow, normal to each panel, due to roll rate is then:

$$\{\alpha_{P_N}\} = \left([Y - Y_{C.G.}]\{n_z\} - [Z - Z_{C.G.}]\{n_y\}\right)\frac{P}{V} \tag{7.25}$$

or

$$\{\alpha_{P_N}\} = \frac{2}{b_{\mathrm{Re}f}}\left([Y - Y_{C.G.}]\{n_z\} - [Z - Z_{C.G.}]\{n_y\}\right)\frac{P\, b_{\mathrm{Re}f}}{2V} \tag{7.26}$$

or

$$\{\alpha_{P_N}\} = \frac{2}{b_{\mathrm{Re}f}}\left([Y - Y_{C.G.}]\{n_z\} - [Z - Z_{C.G.}]\{n_y\}\right)\cdot \hat{p} \tag{7.27}$$

This can be treated just like a camber vector (twisting the airplane). Just as in pitch damping, think of it as curving the airplane rather than the airflow. Equation 7.27 can be added to Equation 3.10 to get:

$$\{\alpha_N\} = \{\alpha_{N_0}\} + \{\alpha_{P_N}\} \tag{7.28}$$

By substituting this equation for $\{\alpha_N\}$ in place of $\{\alpha_{N_0}\}$ in Equation 3.9, the effect of a constant roll rate can be taken into account in computing forces and moments.

It is of interest to compute the change in side force with a change in roll rate angle, $C_{Y_{\hat{p}}}$:

$$C_{Y_{\hat{p}}} = C_{Y_{\hat{p}}}\Big|_{\alpha=0} = \frac{\partial C_Y}{\partial \hat{p}} \tag{7.29}$$

From Equation 5.13, it can be seen that:

$$\frac{\partial C_Y}{\partial \hat{p}} = 2\sum_{n=1}^{N}\left\{\frac{\partial C_{L_N}}{\partial \hat{p}}\right\}\{n_y\} \tag{7.30}$$

From Equation 5.4, it can be seen that:

$$[W]\left\{\frac{\partial C_{L_N}}{\partial \hat{p}}\right\} = \frac{2}{S_{\mathrm{Re}f}}[b]\left\{\frac{\partial \alpha_N}{\partial \hat{p}}\right\} \tag{7.31}$$

Finally, from Equation 7.27 and 7.28, it can be seen that:

$$\left\{\frac{\partial \alpha_N}{\partial \hat{p}}\right\} = \left\{\frac{\partial \alpha_{P_N}}{\partial \hat{p}}\right\} = \frac{2}{b_{\mathrm{Re}f}}\left([Y - Y_{C.G.}]\{n_z\} - [Z - Z_{C.G.}]\{n_y\}\right) \tag{7.32}$$

Again, as with the pitch and yaw damping derivatives, the roll damping derivatives can be calculated using exactly the same process as the longitudinal forces and moments, but with the substitution of Equation 7.32 for Equation 3.10.

The same process is used to compute $C_{Nb_{\hat{p}}}$ and $C_{lb_{\hat{p}}}$.

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